International Journal of Applied Mathematics

Volume 35 No. 5 2022, 693-709
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v35i5.5

A PREDICTOR-CORRECTOR NUMERICAL APPROACH TO EQUATIONS WITH GENERAL FRACTIONAL DERIVATIVE

Ivan Bazhlekov [§], Emilia Bazhlekova

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
Sofia - 1113, BULGARIA

Abstract: The Adams-type predictor-corrector method for the numerical solution of fractional differential equations proposed by K. Diethelm et al. (Nonlinear Dynam. **29** (2002), 3-22) is extended in this work to equations with general fractional derivative. The method may be used both for linear and nonlinear problems. Numerical examples are given for the particular cases of multi-term and distributed-order fractional differential operators, which demonstrate the viability of the developed numerical algorithm.

AMS Subject Classification: 26A33, 65L06, 65R20
Key Words: general fractional derivative; numerical solution; predictor-corrector method; multinomial Mittag-Leffler function; Sonine kernels

1. Introduction

In a series of papers (see e.g. [8, 9]) an Adams-type predictor-corrector algorithm is discussed for the numerical solution of the initial-value problem for the nonlinear fractional differential equation

$$({}^C D_t^\alpha y)(t) = f(t, y(t)), \quad t \in (0, T]. \tag{1}$$

Here ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha > 0$

$$({}^C D_t^\alpha y)(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} y(\tau)\, d\tau,$$

where $m \in \mathbb{N}$ is such that $m - 1 < \alpha \leq m$. Equation (1) is equipped with m

Received: August 15, 2022

[§]Correspondence author

initial conditions, $y^{(k)}(0) = c_k \in \mathbb{R}$, $k = 0, 1, \ldots, m-1$. For $0 < \alpha \le 1$, which is the case of many applications, only one initial condition need to be specified: $y(0) = c_0 \in \mathbb{R}$.

The numerical algorithm developed in [8, 9] is a generalization of the classical Adams-Bashforth-Moulton integration that is well known for the numerical solution of first-order problems. It is based on the equivalent representation of equation (1) as a Volterra integral equation. This algorithm has found applications for the numerical solution of problems in different areas, e.g. [3, 4, 26].

The concept of general fractional calculus (GFC) has recently gained an increasing interest, see e.g. the review article [27] and the references cited there. The idea of GFC was introduced by A. N. Kochubei in [16]. For recent developments we refer to [17, 18, 19].

This work is concerned with the Cauchy problem for the general fractional differential equation

$$({}^{C}\mathbb{D}_t^{(\kappa)} y)(t) = f(t, y(t)), \quad t \in (0, T], \quad y(0) = y_0 \in \mathbb{R}, \tag{2}$$

where ${}^{C}\mathbb{D}_t^{(\kappa)}$ denotes the general fractional derivative (GFD) of Caputo type, which is a generalization of the Caputo fractional derivative ${}^{C}D_t^{\alpha}$, $0 < \alpha < 1$. Other special examples of GFD of Caputo type are the linear combination with positive coefficients of Caputo fractional derivatives of orders in the interval $(0, 1)$ and the distributed-order Caputo fractional derivative with continuous distribution in an interval $\subseteq (0, 1)$. The precise definition of the operator ${}^{C}\mathbb{D}_t^{(\kappa)}$ is given in Section 3.

The general fractional differential equation (2) describes dynamical systems with a general form of nonlocality in time [27].

The aim of the present work is to develop a predictor-corrector algorithm for the numerical solution of the initial-value problem (2) by applying a technique, analogous to that in [8, 9].

The rest of the paper is organized as follows. Section 2 contains preliminaries. In Section 3 the general fractional derivative and the corresponding differential equation (2) are discussed. In Section 4, the predictor and corrector formulae for the general fractional equation are derived. To assess the viability of the presented predictor-corrector algorithm, in Section 5 three numerical examples are considered.

2. Preliminaries

In this work the Laplace transform of a function $f(t)$ is denoted by $\widehat{f}$ or $\mathcal{L}\{f\}$, that is

$$\widehat{f}(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt.$$

Denote as usual by $*$ the Laplace convolution

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau.$$

A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be completely monotone function ($\mathcal{CMF}$) if it is of class C^∞ and

$$(-1)^n f^{(n)}(t) \geq 0, \quad t > 0, \ n = 0, 1, 2, ... \tag{3}$$

The characterization of the class $\mathcal{CMF}$ is given by the Bernstein's theorem: a function is completely monotone if and only if it can be represented as the Laplace transform of a non-negative (generalized) function. The class $\mathcal{CMF}$ is closed under pointwise addition, multiplication and convergence [24].

The class of Stieltjes functions ($\mathcal{SF}$) consists of all functions defined on $\mathbb{R}_+$ which have the representation (see [16])

$$\varphi(s) = \frac{a}{s} + b + \int_0^\infty e^{-s\tau} \psi(\tau)\, d\tau, \quad s > 0,$$

where $a, b \geq 0$ and the locally integrable function $\psi \in \mathcal{CMF}$ is such that the Laplace transform of ψ exists for any $s > 0$.

The Mittag-Leffler function is defined by the series [11]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta + \alpha k)}, \qquad z \in \mathbb{C}, \ \alpha, \beta \in \mathbb{R}, \ \alpha > 0, \tag{4}$$

where $\Gamma(\cdot)$ is the Gamma function.

The following multinomial Mittag-Leffler function is proposed in [12]:

$$E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1, \ldots, z_m) = \sum_{k=0}^\infty \sum_{\substack{k_1+\ldots+k_m=k \\ k_1 \geq 0, \ldots, k_m \geq 0}} \frac{k!}{k_1! \ldots k_m!} \frac{\prod_{j=1}^m z_j^{k_j}}{\Gamma\left(\beta + \sum_{j=1}^m \alpha_j k_j\right)},$$

where $z_j \in \mathbb{C}$, $\alpha_j > 0$, $\beta \in \mathbb{R}$, $j = 1, \ldots, m$.

For convenience we give explicitly the defining series in the binomial case $m = 2$:

$$E_{(\alpha_1,\alpha_2),\,\beta}(z_1,z_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(k_1+k_2)!}{k_1!k_2!} \frac{z_1^{k_1} z_2^{k_2}}{\Gamma\left(\beta+\alpha_1 k_1+\alpha_2 k_2\right)}, \tag{5}$$

where $z_1, z_2 \in \mathbb{C},\ \alpha_1, \alpha_2, \beta \in \mathbb{R}, \alpha_1 > 0, \alpha_2 > 0$.

The Laplace transform pair for the Mittag-Leffler function [11]

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right\}(s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda} \tag{6}$$

is generalized for multinomial Mittag-Leffler type functions as follows (see e.g. [5]):

$$\mathcal{L}\left\{t^{\beta-1}E_{(\alpha_1,\ldots,\alpha_m),\beta}(\lambda_1 t^{\alpha_1},\ldots,\lambda_m t^{\alpha_m})\right\}(s) = \frac{s^{-\beta}}{1-\sum_{j=1}^{m}\lambda_j s^{-\alpha_j}}. \tag{7}$$

A sufficient condition for complete monotonicity of the multinomial Mittag-Leffler type function is the following (see e.g. [5]): If $0 < \alpha_j \le \beta \le 1$ and $\lambda_j < 0$, $j = 1,\ldots,m$, then

$$t^{\beta-1}E_{(\alpha_1,\ldots,\alpha_m),\beta}(\lambda_1 t^{\alpha_1},\ldots,\lambda_m t^{\alpha_m}) \in \mathcal{CMF}, \quad t > 0. \tag{8}$$

Other types of multi-index generalizations of the classical Mittag-Leffler functions and their properties are discussed in [13, 14, 21, 22, 23].

3. Cauchy problem for a nonlinear ODE with the general fractional derivative

Consider the nonlinear general fractional differential equation

$$({}^{C}\mathbb{D}_t^{(\kappa)}y)(t) = f(t,y(t)), \quad t \in (0,T], \tag{9}$$

subject to the initial condition

$$y(0) = y_0 \in \mathbb{R}. \tag{10}$$

Here ${}^{C}\mathbb{D}_t^{(\kappa)}$ denotes the general fractional derivative of the Caputo type, introduced in [16] as follows

$$({}^{C}\mathbb{D}_t^{(\kappa)}f)(t) = \frac{d}{dt}\int_0^t \kappa(t-\tau)f(\tau)\,d\tau - \kappa(t)f(0), \quad t > 0, \tag{11}$$

where $\kappa(t)$ is a nonnegative locally integrable kernel ($\kappa \in L^1_{loc}(\mathbb{R}_+)$).

Definition (11) is applicable for absolutely continuous functions $f \in AC(\overline{\mathbb{R}_+})$. If moreover f' is a locally integrable function then by applying the identity

$$(\kappa * f)'(t) = (\kappa * f')(t) + \kappa(t)f(0) \tag{12}$$

we obtain the following equivalent representation

$$({}^C\mathbb{D}_t^{(\kappa)} f)(t) = (\kappa * f')(t).$$

From here we recognize that ${}^C\mathbb{D}_t^{(\kappa)}$ is indeed a derivative of Caputo type.

As in [16], we make the following assumptions on the kernel $\kappa(t)$ in definition (11). We suppose that the Laplace transform $\widehat{\kappa}(s)$ of the kernel $\kappa(t)$ exists for all $s > 0$ and $\widehat{\kappa}(s) \in \mathcal{SF}$, where $\mathcal{SF}$ is the class of Stieltjes functions. Moreover, we assume

$$\begin{aligned} &\widehat{\kappa}(s) \to 0, \quad s\widehat{\kappa}(s) \to \infty \qquad \text{as} \quad s \to \infty; \\ &\widehat{\kappa}(s) \to \infty, \quad s\widehat{\kappa}(s) \to 0 \qquad \text{as} \quad s \to 0. \end{aligned}$$

The above assumptions on $\widehat{\kappa}(s)$ imply the following properties (see e.g. [16]): $\kappa(t) \in \mathcal{CMF}$ and there exists a function $k(t)$, defined for $t > 0$, which is locally integrable and completely monotone, $k(t) \in L^1_{loc}(\mathbb{R}_+) \cap \mathcal{CMF}$, and the identity is satisfied:

$$(\kappa * k)(t) \equiv 1. \tag{13}$$

The general fractional integral operator $\mathbb{J}_t^{(k)}$ defined by the identity

$$(\mathbb{J}_t^{(k)} f)(t) = \int_0^t k(t-\tau) f(\tau)\, \mathrm{d}\tau$$

satisfies the relations

$${}^C\mathbb{D}_t^{(\kappa)}\, \mathbb{J}_t^{(k)} f(t) = f(t) \quad \text{for any} \quad f \in L^1_{loc}(\mathbb{R}_+), \tag{14}$$

$$\mathbb{J}_t^{(k)}\, {}^C\mathbb{D}_t^{(\kappa)} f(t) = f(t) - f(0) \quad \text{for any} \quad f \in AC(\overline{\mathbb{R}_+}). \tag{15}$$

Pairs of kernels satisfying (13) are referred to as Sonine kernels. In Laplace domain identity (13) reads

$$\widehat{\kappa}(s)\widehat{k}(s) = 1/s. \tag{16}$$

Basic examples of kernels $\kappa(t)$, which obey the required properties, are considered next. The corresponding Sonine kernels $k(t)$ and the Laplace transforms are given. For the sake of brevity we use the notation

$$\omega_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad \beta > 0. \tag{17}$$

The Laplace transform pairs

$$\mathcal{L}\left\{\omega_\beta(t)\right\}(s) = s^{-\beta}, \quad \beta > 0, \tag{18}$$

(6), (7), and the identity (16) are used.

Example 1. *The power-law memory kernel*

$$\kappa(t) = \omega_{1-\alpha}(t), \quad \widehat{\kappa}(s) = s^{\alpha-1}, \qquad 0 < \alpha < 1. \tag{19}$$

In this case ${}^C\mathbb{D}_t^{(\kappa)}$ coincides with the Caputo fractional derivative ${}^C D_t^\alpha$ of order α. The corresponding Sonine kernel k is

$$k(t) = \omega_\alpha(t), \quad \widehat{k}(s) = s^{-\alpha}.$$

Example 2. *The multi-term power-law memory kernel*

$$\kappa(t) = \sum_{j=1}^{m} q_j \omega_{1-\alpha_j}(t), \quad \widehat{\kappa}(s) = \sum_{j=1}^{m} q_j s^{\alpha_j - 1}, \tag{20}$$

where $1 > \alpha_1 > \alpha_2 > \ldots > \alpha_m > 0$, $q_j > 0$, $j = 1, \ldots, m$, $m > 1$. In this case ${}^C\mathbb{D}_t^{(\kappa)}$ is a linear combination of fractional derivatives of Caputo type:

$${}^C\mathbb{D}_t^{(\kappa)} = \sum_{j=1}^{m} q_j \, {}^C D_t^{\alpha_j}.$$

Without loss of generality we may assume $q_1 = 1$. In this case (16) implies

$$\widehat{k}(s) = \frac{1}{\sum_{j=1}^{m} q_j s^{\alpha_j}} = \frac{s^{-\alpha_1}}{1 + \sum_{j=2}^{m} q_j s^{-(\alpha_1 - \alpha_j)}}. \tag{21}$$

Therefore, taking into account (7), we derive a representation of $k(t)$ as a multinomial Mittag-Leffler type function

$$k(t) = t^{\alpha_1 - 1} E_{(\alpha_1 - \alpha_2, \ldots, \alpha_1 - \alpha_m), \alpha_1}\left(-q_2 t^{\alpha_1 - \alpha_2}, \ldots, -q_m t^{\alpha_1 - \alpha_m}\right).$$

Let us note that this function satisfies the sufficient conditions for complete monotonicity and (8) implies $k(t) \in \mathcal{CMF}$.

In particular, in the two-term case ($m = 2$)

$$\kappa(t) = \omega_{1-\alpha_1}(t) + q\omega_{1-\alpha_2}(t), \quad \widehat{\kappa}(s) = s^{\alpha_1 - 1} + q s^{\alpha_2 - 1}, \tag{22}$$

where $1 > \alpha_1 > \alpha_2 > 0$, $\quad q > 0$. Applying relation (16) it follows $\widehat{k}(s) = s^{-\alpha_2}/(s^{\alpha_1 - \alpha_2} + q)$ and therefore (6) implies

$$k(t) = t^{\alpha_1 - 1} E_{\alpha_1 - \alpha_2, \alpha_1}(-q t^{\alpha_1 - \alpha_2}). \tag{23}$$

Example 3. *The distributed-order memory kernel*

$$\kappa(t) = \int_0^1 \omega_{1-\alpha}(t)\mu(\alpha)\, d\alpha, \quad \widehat{\kappa}(s) = \int_0^1 s^{\alpha-1}\mu(\alpha)\, d\alpha, \tag{24}$$

where $\mu(\cdot)$ is a nonnegative weight function. In this case ${}^C\mathbb{D}_t^{(\kappa)}$ is the Caputo fractional derivative of distributed order:

$${}^C\mathbb{D}_t^{(\kappa)} = \int_0^1 \mu(\alpha) {}^C D_t^\alpha \, d\alpha.$$

In the particular case of uniform distribution, $\mu \equiv 1$, the memory kernel becomes

$$\kappa(t) = \int_0^1 \omega_{1-\alpha}(t)\, d\alpha, \qquad \widehat{\kappa}(s) = \int_0^1 s^{\alpha-1}\, d\alpha = \frac{s-1}{s\log s}. \tag{25}$$

Therefore, according to (16), the Sonine kernel $k(t)$ satisfies

$$\widehat{k}(s) = \frac{\log s}{s-1}.$$

By applying the Titchmarsh theorem for the inverse Laplace transform we obtain the integral representation for the kernel $k(t)$

$$k(t) = \int_0^\infty e^{-rt} K(r)\, dr, \tag{26}$$

where

$$K(r) = -\frac{1}{\pi}\Im\left\{ \left.\frac{\log s}{s-1}\right|_{s=re^{i\pi}} \right\} = \frac{1}{r+1}.$$

Let us note that, since $K(r) > 0$ for $r \geq 0$, representation (26) implies $k(t) \in \mathcal{CMF}$. Plugging $K(r)$ in (26) yields

$$k(t) = \int_0^\infty \frac{e^{-rt}}{r+1}\, dr = e^t \mathbb{E}_1(t), \tag{27}$$

where $\mathbb{E}_1(t)$ denotes the exponential integral [1]

$$\mathbb{E}_1(t) = \int_t^\infty \frac{e^{-\xi}}{\xi}\, d\xi.$$

The general case of kernels (24) with arbitrary continuous weight function $\mu(\alpha)$ is studied in [15].

Existence and uniqueness of solution to the Cauchy problem (9)-(10) is studied in [25] under appropriate conditions on the function f, see also the review paper [20] for a summary of the established existence and uniqueness results.

In this work we suppose that the conditions for the existence of a unique solution are satisfied. Our aim is to present an algorithm for the numerical computation of this solution.

4. The predictor-corrector algorithm

Let us start with the predictor-corrector method, proposed in [8, 9], for the numerical solution of the initial-value problem (9)-(10), in the particular case when ${}^C\mathbb{D}_t^{(\kappa)}$ is the Caputo fractional derivative ${}^C D_t^\alpha$ of order α (the case of kernel (19)). To deduce the predictor and corrector formulae, in [8, 9] the equivalent representation of the problem as a Volterra integral equation is used

$$y(t) = y_0 + \int_0^t \omega_\alpha(t-\tau) f(\tau, y(\tau))\, d\tau, \quad t \in (0, T]. \tag{28}$$

The integral equation (28) is solved on a uniform mesh $t_j = jh$, $h = T/N$, $j = 0, 1, ..., N$, in two steps: predictor and corrector step. To approximate the integral in (28), the product rectangle rule is used in the predictor step and the product trapezoidal quadrature formula in the corrector step. In this way the following algorithm is derived, where y_j denotes the approximation for $y(t_j)$:

Predictor step:

$$y_{n+1}^P = y_0 + \sum_{j=0}^{n} b_{j,n+1}\, f(t_j, y_j), \tag{29}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\Gamma(\alpha+1)}((n+1-j)^\alpha - (n-j)^\alpha). \tag{30}$$

Corrector step:

$$y_{n+1} = y_0 + \sum_{j=0}^{n} a_{j,n+1}\, f(t_j, y_j) + a_{n+1,n+1}\, f(t_{n+1}, y_{n+1}^P), \tag{31}$$

where

$$a_{j,n+1} = \frac{h^\alpha}{\Gamma(\alpha+2)} A_{j,n+1} \tag{32}$$

and the coefficients $A_{j,n+1}$ are defined by

$$A_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha} & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} & \\ \qquad\qquad -2(n-j+1)^{\alpha+1} & \text{if } 1 \le j \le n, \\ 1 \quad \text{if } j = n+1. & \end{cases}$$

Applying an analogous argument, next we generalize the above algorithm to the case of the equation with general fractional derivative (9).

The following Volterra integral equation, equivalent to (9), is obtained by the use of (14) and (15) (see also Lemma 2.1 in [25]):

$$y(t) = y_0 + \int_0^t k(t-\tau) f(\tau, y(\tau))\, d\tau, \quad t \in (0, T], \tag{33}$$

where the kernel $k(t)$ is the Sonine kernel to $\kappa(t)$, defined by the identity (13).

Let us denote

$$k_1(t) = \int_0^t k(\tau)\, d\tau, \quad k_2(t) = \int_0^t k_1(\tau)\, d\tau. \tag{34}$$

Applying the product rectangle rule, we derive for the integral in (33)

$$\int_0^{t_{n+1}} k(t_{n+1} - \tau) g(\tau)\, d\tau \approx \sum_{j=0}^{n} b_{j,n+1} g(t_j),$$

where

$$b_{j,n+1} = \int_{t_j}^{t_{j+1}} k(t_{n+1} - \tau)\, d\tau = k_1(t_{n+1} - t_j) - k_1(t_{n+1} - t_{j+1}).$$

Therefore, in the case of general fractional derivative the coefficients $b_{j,n+1}$ in the predictor formula (29) have the form

$$b_{j,n+1} = k_1((n+1-j)h) - k_1((n-j)h). \tag{35}$$

For the corrector step we use the product trapezoidal quadrature formula with respect to the weight function $k(t_{n+1} - \cdot)$ to replace the integral. This implies by using standard techniques from quadrature theory (see e.g. [7])

$$\int_0^{t_{n+1}} k(t_{n+1} - \tau) g(\tau)\, d\tau \approx \sum_{j=0}^{n+1} a_{j,n+1} g(t_j),$$

where

$$a_{j,n+1} = \int_0^{t_{n+1}} k(t_{n+1}-\tau)\phi_{j,n+1}(\tau)\,d\tau$$

with

$$\phi_{j,n+1}(\tau) = \begin{cases} \dfrac{\tau - t_{j-1}}{t_j - t_{j-1}}, & \tau \in (t_{j-1}, t_j], \\ \dfrac{t_{j+1}-\tau}{t_{j+1}-t_j}, & \tau \in (t_j, t_{j+1}), \\ 0, & \text{otherwise.} \end{cases}$$

In this way we obtain for the case of general fractional derivative the following representations of the coefficients $a_{j,n+1}$ in the corrector formula (31):

$$a_{j,n+1} = \begin{cases} k_1((n+1)h) + \dfrac{k_2(nh) - k_2((n+1)h)}{h}, & j = 0, \\ \dfrac{k_2((n+2-j)h) + k_2((n-j)h)}{h} \\ \qquad\qquad -2\dfrac{k_2((n+1-j)h)}{h}, & 1 \le j \le n, \\ \dfrac{k_2(h)}{h}, \quad j = n+1. \end{cases} \tag{36}$$

As a final result, we deduced the following two-step algorithm for the numerical computation of the solution to the initial-value problem with the GFD (9)-(10):

Predictor step: predictor formula (29) with coefficients $b_{j,n+1}$, given in (35);

Corrector step: corrector formula (31) with coefficients $a_{j,n+1}$, given in (36).

The functions k_1 and k_2 are defined in (34), where $k(t)$ is the Sonine kernel of the kernel $\kappa(t)$ in the definition of the GFD.

It is not difficult to check that in the case of single Caputo derivative ${}^C\mathbb{D}_t^{(\kappa)} = {}^CD_t^\alpha$, which corresponds to $k_1(t) = \omega_{\alpha+1}(t)$ and $k_2(t) = \omega_{\alpha+2}(t)$, formulae (35) and (36) reduce to (30) and (32), respectively.

The deduced predictor-corrector algorithm is applied to some numerical examples in the next section.

5. Numerical examples

To evaluate numerically the presented technique, we consider three examples, in which we have either an exact solution, or a known integral representation of the solution.

Example 4. *A nonlinear equation with two Caputo fractional derivatives of different orders.*

Let $1 > \alpha_1 > \alpha_2 > 0$, $q > 0$, and consider the following nonlinear initial-value problem

$$ {}^C D_t^{\alpha_1} y(t) + q\, {}^C D_t^{\alpha_2} y(t) = f_{\alpha_1,\alpha_2}(t, y(t)), \quad y(0) = 0, \tag{37} $$

where the function $f_{\alpha_1,\alpha_2}(t, y)$ is defined as follows

$$ f_{\alpha_1,\alpha_2}(t, y) = \varphi_{\alpha_1,\alpha_1}(t) + q\varphi_{\alpha_1,\alpha_2}(t) + \left(1.5t^{\alpha_1/2} - t^4\right)^3 - y^{3/2} $$

with

$$ \varphi_{\alpha,\beta}(t) = \frac{40320}{\Gamma(9-\beta)} t^{8-\beta} - \frac{3\Gamma(5+\alpha/2)}{\Gamma(5+\alpha/2-\beta)} t^{4+\alpha/2-\beta} + \frac{2.25\Gamma(\alpha+1)}{\Gamma(\alpha+1-\beta)} t^{\alpha-\beta}. $$

The exact solution of the initial-value problem (37) is

$$ y(t) = t^8 - 3t^{4+\alpha_1/2} + 2.25t^{\alpha_1}. \tag{38} $$

We solve numerically (37) for $\alpha_1 = 0.8$, $\alpha_2 = 0.5$, and $q = 1$, by applying the predictor-corrector algorithm described in the previous section. The functions k_1 and k_2 are defined in (34), where the kernel $k(t)$ is given by the Mittag-Leffler function (23). Based on formula (A.32) in [10] for the Mittag-Leffler type function, for the computation of $k_1(t)$ we obtain and use the integral formula (see also [6], Eq.(92))

$$ k_1(t) = \frac{1}{\pi} \int_0^\infty \frac{1 - e^{-rt}}{r^{\alpha_2+1}} K_{\alpha_1,\alpha_2}(r)\, dr, \tag{39} $$

where

$$ K_{\alpha_1,\alpha_2}(r) = \frac{\sin \alpha_2\pi + r^{\alpha_1-\alpha_2} \sin \alpha_1\pi}{r^{2(\alpha_1-\alpha_2)} + 2r^{\alpha_1-\alpha_2} \cos(\alpha_1 - \alpha_2)\pi + 1}. $$

The function $k_2(t)$ is evaluated by numerical integration of $k_1(t)$.

Information about the error in the numerical computation of solution is given in Table 1.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
err	3.4×10^{-3}	2.6×10^{-4}	1.5×10^{-5}	7.8×10^{-7}	$1. \times 10^{-7}$

Table 1: The difference between the numerical solution of problem (37) and the exact solution (38) for $\alpha_1 = 0.8$, $\alpha_2 = 0.5$, and $q = 1$ at $t = 1.0$.

Example 5. *A linear inhomogeneous equation with two Caputo fractional derivatives of different orders.*

Let $1 > \alpha_1 > \alpha_2 > 0$, $q, \lambda > 0$, and consider the linear equation

$$ {}^C D_t^{\alpha_1} y(t) + q\, {}^C D_t^{\alpha_2} y(t) = 1 - \lambda y(t), \quad y(0) = 0. \tag{40}$$

The solution $y(t)$ is given by the following binomial Mittag-Leffler function [5, 6]:

$$ y(t) = t^{\alpha_1} E_{(\alpha_1, \alpha_1 - \alpha_2), \alpha_1 + 1} \left(-\lambda t^{\alpha_1}, -q t^{\alpha_1 - \alpha_2} \right). $$

Moreover, the solution admits the following integral representation, which can be used for numerical computation (cf. [6], Theorem 7.3)

$$ y(t) = \frac{1}{\lambda} - \frac{1}{\pi} \int_0^\infty \frac{e^{-rt}}{r} \cdot \frac{I(r)}{(R(r) + \lambda)^2 + (I(r))^2}\, dr, \tag{41}$$

where

$$ \begin{aligned} R(r) &= r^{\alpha_1} \cos \alpha_1 \pi + q r^{\alpha_2} \cos \alpha_2 \pi, \\ I(r) &= r^{\alpha_1} \sin \alpha_1 \pi + q r^{\alpha_2} \sin \alpha_2 \pi. \end{aligned} $$

Numerical computation of the solution of (40) for $\alpha_1 = 0.8$, $\alpha_2 = 0.5$, $q = 1$, and $\lambda = 1, 2, 3$, is performed by using the predictor-corrector algorithm with kernels $k_1(t)$ and $k_2(t)$ from the first numerical example and, alternatively, by applying the integral representation (41). A comparison between the two numerical solutions is presented in Figure 1. The predictor-corrector algorithm is performed with time step $h = 10^{-3}$. The difference between the two solutions is of the order of 10^{-7}.

Example 6. *Relaxation function for the equation with uniformly distributed order.*

Consider the linear relaxation equation of distributed order with uniform distribution

$$ \int_0^1 {}^C D_t^{\alpha} y(t)\, d\alpha = -\lambda y(t), \quad y(0) = 1, \tag{42}$$

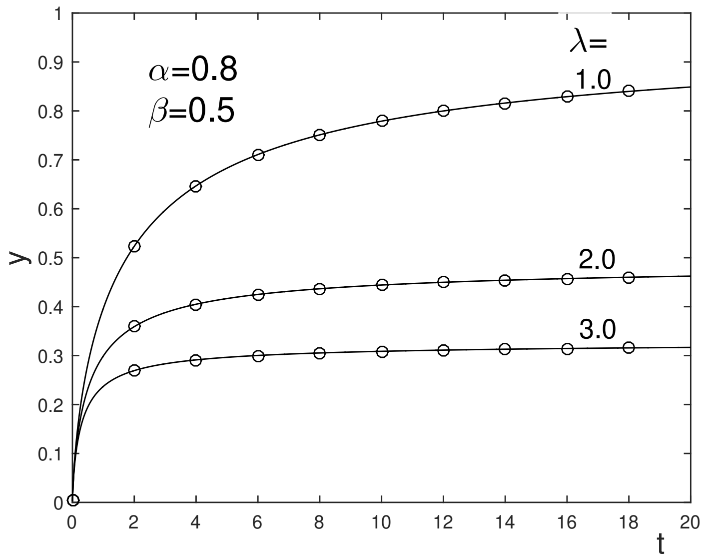

Figure 1: Solution $y(t)$ of problem (40) for $\alpha_1 = 0.8$, $\alpha_2 = 0.5$, $q = 1$, and $\lambda = 1, 2, 3$. The numerical solution obtained by predictor-corrector algorithm is given by solid lines and the marks $\circ$ correspond to that given by (41).

where $\lambda > 0$. The solution $y(t)$ admits the integral representation (see e.g. [2])

$$y(t) = \int_0^\infty e^{-rt} K_D(r)\, dr, \tag{43}$$

where

$$K_D(r) = \frac{1}{\pi r} \frac{\lambda B(r)}{(A(r) + \lambda)^2 + (B(r))^2} \tag{44}$$

with

$$A(r) = -\frac{(r+1)\log r}{\log^2 r + \pi^2}, \quad B(r) = \frac{\pi(r+1)}{\log^2 r + \pi^2}.$$

Numerical computation of the solution $y(t)$ of initial-value problem (42) is performed in two ways: by using the integral representation (43)-(44) and by applying the predictor-corrector method, where the functions $k_1(t)$ and $k_2(t)$ are found by numerical integration of the kernel $k(t)$, which for this case is given by (27). A comparison between the two solutions is given in Figure 2. The predictor-corrector algorithm is performed with time step $h = 10^{-3}$. The difference between the two solutions is of the order of 10^{-2} for $\lambda = 1$ and 10^{-3} for $\lambda = 2$ and $\lambda = 3$. Compared to the previous two examples, the accuracy is not so high, which is mainly due to the error in calculation of the infinite integral in (43).

Let us note that the infinite integrals in equations (39), (41) and (43) are calculated numerically using Matlab.

The presented numerical experiments demonstrate that the developed predictor-corrector numerical algorithm is useful and viable and can be further studied.

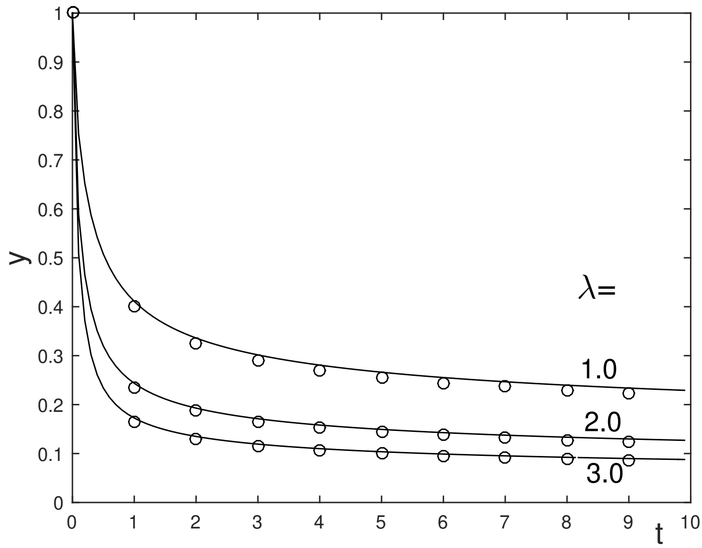

Figure 2: Solution $y(t)$ of problem (42) for $\lambda = 1, 2, 3$. The numerical solution obtained by predictor-corrector algorithm is given by solid lines and the marks $\circ$ correspond to that given by the integral representation (43)-(44).

Acknowledgements

The first author (I.B.) is supported by the Bulgarian National Science Fund under Grant FNI KP-06-H22/2.

The second author (E.B.) is supported by the National Scientific Program “Information and Communication Technologies for a Single Digital Market in

Science, Education and Security (ICTinSES)", contract No DO1–205/23.11.2018, financed by the Ministry of Education and Science in Bulgaria.

This work is performed in the frames of the Bilateral Research Project "Operators, differential equations and special functions of Fractional Calculus – numerics and applications" between Bulgarian Academy of Sciences and Serbian Academy of Sciences and Arts.

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* Dover, New York (1964).

[2] E. Bazhlekova, I. Bazhlekov, Application of Dimovski's convolutional calculus to distributed-order time-fractional diffusion equation on a bounded domain, *J. of Inequalities and Special Functions*, **8** (2017), 68–83.

[3] I. Bazhlekov, E. Bazhlekova, Fractional derivative model for diffusion-controlled adsorption at liquid/liquid interface, *AIP Conf. Proc.*, **2048** (2018), Art. 050012.

[4] I. Bazhlekov, E. Bazhlekova, Fractional derivative modeling of bioreaction-diffusion processes, *AIP Conf. Proc.*, **2333** (2020), Art. 060006.

[5] E. Bazhlekova, Completely monotone multinomial Mittag-Leffler type functions and diffusion equations with multiple time-derivatives, *Fract. Calc. Appl. Anal.*, **24**, No 1 (2021), 88–111; https://doi.org/10.1515/fca-2021-0005.

[6] E. Bazhlekova, I. Bazhlekov, Identification of a space-dependent source term in a nonlocal problem for the general time-fractional diffusion equation. *J. Comput. Appl. Math.* **386** (2021), Art. 113213.

[7] K. Diethelm, A. Freed, The FracPECE subroutine for the numerical solution of differential equations of fractional order, In: *Forschung und wissenschaftliches Rechnen: Beiträge zum Heinz-Billing-Preis 1998*, Eds. S. Heinzel and T. Plesser (Gesellschaft für wissenschaftliche Datenverarbeitung, Göttingen, 1999), 57–71.

[8] K. Diethelm, N. Ford, A. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, *Nonlinear Dynam.*, **29** (2002), 3–22.

[9] K. Diethelm, N. Ford, A. Freed, Detailed error analysis for a fractional Adams method, *Numer. Algorithms*, **36** (2004), 31–52.

[10] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order. In: *Fractals and Fractional Calculus in Continuum Mechanics*, Eds.: A. Carpinteri, F. Mainardi, Springer-Verlag, Wien-New York (1997), 223–276.

[11] R. Gorenflo, A. Kilbas, F. Mainardi, S. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications,* 2nd Ed., Springer, Berlin-Heidelberg (2020).

[12] S.B. Hadid, Y. Luchko, An operational method for solving fractional differential equations of an arbitrary real order. *Panam. Math. J.*, **6**, No 1 (1996), 57–73.

[13] V. Kiryakova, Unified approach to fractional calculus images of special functions - a survey, *Mathematics*, **8**, No 12 (2020), Art. 2260.

[14] V. Kiryakova, A guide to special functions in fractional calculus, *Mathematics*, **9**, No 1 (2021), Art. 106.

[15] A.N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.*, **340** (2008), 252–281.

[16] A.N. Kochubei, General fractional calculus, evolution equations, and renewal processes, *Integral Equ. Oper. Theory*, **71** (2011), 583–600.

[17] Yu. Luchko, General fractional integrals and derivatives of arbitrary order, *Symmetry*, **13** (2021), 755.

[18] Yu. Luchko, Operational calculus for the general fractional derivative and its applications, *Fract. Calc. Appl. Anal.*, **24**, No 2 (2021), 338–375; https://doi.org/10.1515/fca-2021-0016.

[19] Yu. Luchko, General fractional integrals and derivatives with the Sonine kernels, *Mathematics*, **9** (2021), Art. 594.

[20] Yu. Luchko, M. Yamamoto, The general fractional derivative and related fractional differential equations, *Mathematics*, **8** (2020), Art. 2115.

[21] J. Paneva-Konovska, A survey on Bessel type functions as multi-index Mittag-Leffler functions: Differential and integral relations, *Int. J. Appl. Math.* **32**, No 3 (2019), 357–380; doi: 10.12732/ijam.v32i3.1.

[22] J. Paneva-Konovska, Series in Le Roy Type functions: inequalities and convergence theorems, *Int. J. Appl. Math.* **33**, No 6 (2020), 995–1007; doi: 10.12732/ijam.v33i6.3.

[23] J. Paneva-Konovska, V. Kiryakova, On the multi-index Mittag-Leffler functions and their Mellin transforms, *Int. J. Appl. Math.* **33**, No 4 (2020), 549–571; doi: 10.12732/ijam.v33i4.1.

[24] R. Schilling, R. Song, Z. Vondraček, *Bernstein Functions: Theory and Applications*, De Gruyter, Berlin (2010).

[25] C.-S. Sin, Well-posedness of general Caputo-type fractional differential equations, *Fract. Calc. Appl. Anal.*, **21**, No 3 (2018), 819–832; https://doi.org/10.1515/fca-2018-0043.

[26] V.R. Suvandzhieva, Mathematical modeling of bioprocesses with the use of fractional order derivatives, *Biomath Communications*, **8** (2021), 1–48.

[27] V.E. Tarasov, General fractional dynamics, *Mathematics*, **9** (2021), Art. 1464.